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Abstract Cauchy Problems That Involve a Product of Two Euler–Poisson–Darboux Operators

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Let X be a Banach space, let B be the generator of a continuous group in X, and let $A = B^2$. Assume that $\mathscr{D}(A^r)$ is dense in X for r an arbitrarily large positive integer and that a and b are non-negative reals. Solution representations are developed for the abstract differential equation

$$\left(D_t^2 + \frac{b}{t} D_t - A\right) \cdot \left(D_t^2 + \frac{a}{t} D_t - A\right) u(t) = 0, \qquad t > 0$$

corresponding to initial conditions of the form: (i) $u(0+) = \phi$, $u^{(j)}(0+) = 0$, $j = 1, 2, 3$ and (ii) $u^2(0+) = \phi$, $u^j(0+) = 0$, $j = 0, 1, 3$ (with $\varphi \in \mathscr{D}(A^r)$) for all choices of a and b.

1. Introduction

Let X be a Banach space, let B be the generator of a continuous group in X and let $A = B^2$.Then A is the infinitesimal generator of a holomorphic semi-group in X ([8], also see [3]). Assume that the domain $\mathscr{D}(A^r)$ is dense in X for r an arbitrarily large positive integer. We shall be concerned with developing solution representations for the abstract differential equation

$$\left(D_t^2 + \frac{b}{t} D_t - A\right)\left(D_t^2 + \frac{a}{t} D_t - A\right) u(t) = 0, \qquad t > 0 \tag{1.1}$$

subject to the following two types of initial conditions:

$$u(0+) = u'(0+) = u'''(0+) = 0, \qquad u''(0+) = \phi, \quad \phi \in \mathscr{D}(A^r); \tag{1.2a}$$

$$u(0+) = \phi, u'(0+) = u''(0+) = u'''(0+) = 0, \tag{1.2b}$$

In these, the parameters a and b are taken to be non-negative. It is understood that the initial conditions in (1.2) are taken on in the sense of the strong norm, i.e., $u(0+) = \phi$ means that $\|u(t) - \phi\|_{t \to 0+} \to 0$, etc. Solutions of

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(1.1)–(1.2) will be expressed in terms of series and/or integrals of solutions of abstract Euler–Poisson–Darboux problems of the form

$$\left(D_t^2 + \frac{c}{t} D_t - A\right) E(t) = 0, \qquad t > 0, \tag{1.3a}$$

$$E(0+) = \psi, \qquad E'(0+) = 0; \qquad \psi \in \mathscr{D}(A^r) \tag{1.3b}$$

in which c will be related to the parameter a and in which ψ will be related to ϕ. For future reference, we denote the solution of (1.3) by $E^c(t; \psi)$. Our treatment will include all the cases (I) $(a-b)/2$ a non-negative integer, (II) $(a-b)/2 > -1$ but non-integer, (III) $(a-b)/2$ a negative integer, and (IV) $(a-b)/2 < -1$ but non-integer.

There are a number of reasons for our interest in this problem. In their studies on axially symmetric potential theory, Weinstein [12, 13] and Burns [5, 6] developed representations of solutions of equations of the form

$$\left(D_t^2 + \frac{a}{t} D_t \pm \Delta_n\right)^m \cdot u(x, t) = 0, \qquad m = 2, 3, \ldots$$

in which Δ_n denotes the n-dimensional Laplacian operator. The results we obtain provide generalizations of these solution representations in the case $m = 2$. The factorization and reduction procedure we use touches upon control problem methods. Finally, a detailed treatment of (1.1)–(1.2) should provide some useful insights into approaches for solving a variety of Cauchy and Dirichlet type problems in which the differential operator in the underlying equation can be factored into a product of non-commutative operators. It will be seen that the use of relatedness methods [1–4] will require our making use of a variety of tools and techniques of mathematical analysis. In [1], the author briefly discussed this approach for (1.1) (with $A = \Delta_n$) corresponding to a condition of type (1.2a) for the special cases $(a-b)/2 = 0, 1, 2$, and $-\nu$, $0 < \nu < 1$. Thus, our results will generalize and substantially extend these earlier ones.

The essential mathematical background for our treatment will be given in Section 2. This will include a summary of results from [1], a version of the Leibnitz rule in which one of the factors is the solution of an abstract heat problem, and some operational formulas for Laplace transforms. The basic representations for solutions of (1.1)–(1.2) will be stated as theorems in Section 3 and the derivation of the first two of them (corresponding to cases I and II) will be given in Section 4 and 5. Detailed developments of solutions in cases III and IV will not be given since they make use of the methods for treating case II. In Section 6, we will point out the essential differences in initiating these derivations and leave it to the reader to fill in the details.

Throughout this paper, integrals will be taken in the strong Riemann sense to permit the removal of operators from under signs of integration. The choiceof r in $D(A^r)$ will be seen to depend upon the size of $(a-b)/2$ and cannot be specified in advance. We will simply assume that it is large enough to carry out the necessary arguments. Finally, the requirement that a and b be non-negative has been imposed to assure uniqueness of solutions.

2. Basic Preliminaries

We now recall a number of key points from [1] that are relevant to the problems (1.1)–(1.2). Define the function $v(t)$ by the relation

$$\left(D_t^2 + \frac{a}{t} D_t - A\right) u(t) = v(t). \tag{2.1}$$

Then $v(t)$ must satisfy the equation

$$\left(D_t^2 + \frac{b}{t} D_t - A\right) v(t) = 0. \tag{2.2}$$

It is easy to check that the conditions on $v(t)$ that correspond, respectively, to conditions (1.2) are

$$v(0+) = (a+1)\phi, \qquad v'(0+) = 0, \tag{2.3a}$$

$$v(0+) = -A\phi, \qquad v'(0+) = 0. \tag{2.3b}$$

Suppose we let $U(t;\psi)$ denote a solution of the non-homogeneous equation (2.1) that corresponds to the conditions $v(0+) = \psi$, $v'(0+) = 0$, $U(0+;\psi) = U'(0+;\psi) = 0$. It follows that the solutions of (1.1)–(1.2) are given, respectively, by

$$u(t) = \begin{cases} U(t; (a+1)\phi), & (2.4a) \\ E^a(t;\phi) + U(t; -A\phi). & (2.4b) \end{cases}$$

This reduces the construction of solutions of (1.1)–(1.2) to the one of constructing $U(t;\psi)$ with $\psi = (a+1)\phi$ in (2.4a) and $\psi = -A\phi$ in (2.4b).

Now the solution $v(t)$ of (2.2) corresponding to the conditions $v(0+) = \psi$, $v'(0+) = 0$ is given by

$$v(t) = t^{1-b} \Gamma\left(\frac{b+1}{2}\right) \mathscr{L}_s^{-1}\{s^{-(b+1)/2} W(1/4s)\}_{s \to t^2} \tag{2.5}$$

(i.e. $v(t) = E^b(t)$) in which $W(t)$ is a solution of the abstract heat problem

$$\begin{aligned} W'(t) - AW(t) &= 0, \qquad t > 0, \\ W(0+) &= \psi, \qquad \psi \in \mathscr{D}(A^r). \end{aligned} \tag{2.6}$$

In (2.5), $\mathscr{L}_{s}^{-1}\{\ \}_{s \to t^2}$ denotes the inverse Laplace transform with s the variable of the transform and t^2 the variable of inversion (see [2, p. 264]). With the change of variables $\xi = t^2/4$, (2.1) transforms into

$$\left\{ \xi D_\xi \left(\xi D_\xi + \frac{a+1}{2} - 1 \right) - \xi A \right\} u(2\xi^{1/2}) = \xi v(2\xi^{1/2}). \tag{2.7}$$

An equation related to this has the form

$$\{\xi D_\xi - \xi A\}\, Z(2\xi^{1/2}) = f(2\xi^{1/2}) \tag{2.8}$$

in which the functions Z and f are to be determined. By theorem 3.2 of [1], we have

$$U(\xi^{1/2}, \psi) = \xi^{1-(a+1)/2} \Gamma\left(\frac{a+1}{2}\right) \mathscr{L}_s^{-1}\{s^{-(a+1)/2} Z(2s^{-1/2})\}_{s \to \xi} \tag{2.9}$$

provided that f is defined by the relation

$$\xi v(2\xi^{1/2}) = \xi^{2-(a+1)/2} \Gamma\left(\frac{a+1}{2}\right) \mathscr{L}_s^{-1}\{s^{-(a+1)/2} f(2s^{-1/2})\}_{s \to \xi}. \tag{2.10}$$

Finally, if we use the formula (2.5) to replace the function v in (2.10) and then take a Laplace transform, we obtain

$$f(2s^{-1/2}) = \frac{\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)} s^{(a-1)/2} \times \int_0^\infty e^{-s\xi} \xi^{(a-b)/2} \mathscr{L}_s^{-1}\{s^{-(b+1)/2} W(1/s)\}_{s \to \xi}\, d\xi. \tag{2.11}$$

From these relationships, we observe that the following steps are needed to obtain $U(t; \psi)$: (S_1) use (2.11) to obtain the function f, (S_2) introduce this function into (2.8) to determine the function $Z(2\xi^{1/2})$, and (S_3) apply (2.9) to recover $U(t; \psi)$. The computations needed to determine f involve a number of technicalities, the precise ones depending upon the case associated with $(a-b)/2$ (non-negative integer, etc.). Step S_2 requires solving a non-homogeneous abstract heat problem in which the non-homogeneous term itself involves the solution of a homogeneous heat problem.

We now give a number of elementary lemmas that will be called upon later in the construction steps referred to above.

LEMMA 2.1. *Let $W(t)$ be a solution of* (2.6). *Then*

$$(-D_s)^m\{s^{-\beta}W(1/s)\}$$
$$= \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(\beta+m)}{\Gamma(\beta+m-k)} s^{-(\beta+2m-k)} A^{m-k} W(1/s). \qquad (2.12)$$

Proof. This can be derived by using methods given in [10] (see p. 34). However, its validity is easily checked by carrying out an induction on m.

The following two lemmas are restatements of ones given in [1].

LEMMA 2.2. *Let a be a positive parameter. Then*

$$\mathscr{L}_s^{-1}(g(s)\,W(a/s)\}_{s\to\xi} = (4a)\,\mathscr{L}_s^{-1}\{g(4as)\,W(1/4s)\}_{s\to 4a\xi}$$

if either of these inverse Laplace transforms exist.

LEMMA 2.3. *Let $W(\xi)$ be a solution of the abstract heat equation $W_\xi(\xi) - AW(\xi) = 0$ and let a be a positive parameter. Then the solution of the nonhomogeneous equation $Z_\xi - AZ = h(\xi)\,W(a\xi)$ with $h(\xi)$ continuous and $Z(0+) = 0$ is given by*

$$Z(\xi) = \int_0^\xi h(\eta)\,W(a\eta + \xi - \eta)\,d\eta.$$

If $a = 1$, the right-hand member of this reduces to $H(\xi)\,W(\xi)$ with $H(\xi) = \int_0^\xi h(\eta)\,d\eta$.

On a number of occasions, we will find it necessary to compute the inverse Laplace transform of an integral in which the integrand is itself a Laplace transform. The following result will suffice for our purposes:

LEMMA 2.4. *Let $f(x, y)$ be continuous for $0 < x < \infty$ and $a < y < b$ and let $F(s, y) = \int_0^\infty e^{-sx}f(x, y)\,dx$, $s \geqslant c > 0$. If both $\int_a^b(\int_0^\infty e^{-sx}\,|f(x, y)|\,dx)\,dy$ and $\int_0^\infty e^{-sx}(\int_a^b |f(x, y)|\,dy)\,dx$ exist, then*

$$\mathscr{L}_s^{-1}\left\{\int_a^b F(s, y)\,dy\right\}_{s\to x^*} = \int_a^b f(x^*, y)\,dy$$

for $x^* > 0$.

Proof. Now

$$\mathscr{L}_s^{-1}\left\{\int_a^b F(s, y)\,dy\right\}_{s\to x^*}$$
$$=\mathscr{L}_s^{-1}\left\{\int_a^b\left[\int_0^\infty e^{-sx}f(x, y)\,dx\right]dy\right\}_{s\to x^*}$$
$$=\mathscr{L}_s^{-1}\left\{\int_0^\infty e^{-sx}\left[\int_a^b f(x, y)\,dy\right]dx\right\}_{s\to x^*}$$

and this last inverse transform reduces to $\int_a^b f(x^*, y)\,dy$. The integrability conditions contained in the hypotheses were used to permit the interchange of orders of integration (see [7, p. 497]).

Note 1. This lemma can be extended to apply to multiple integrals. If $f(x, y)\in X$, then we replace the absolute value sign in this lemma by $\| \ \|$.

LEMMA 2.5. *Let A be the infinitesimal generator of a holomorphic semigroup $T(t)$. If $0<\nu<1$, $p\geqslant \nu+1/2$, and $s\geqslant c>0$ for some c, then*

$$\int_0^\infty e^{-s\xi}\xi^{-\nu}\mathscr{L}_s^{-1}\{s^{-p}W(1/s)\}_{s\to\xi}\,d\xi$$
$$=\frac{1}{\Gamma(\nu)}\int_0^1 \lambda^{p-\nu-1}(1-\lambda)^{\nu-1}s^{-(p-\nu)}W(\lambda/s)\,d\lambda. \tag{2.13}$$

Proof. The result will follow if the inverse Laplace transforms of both members of (2.13) can be shown to be the same. This involves showing that

$$\xi^{-\nu}\mathscr{L}_s^{-1}\{s^{-p}W(1/s)\}_{s\to\xi}$$
$$=\frac{1}{\Gamma(\nu)}\mathscr{L}_s^{-1}\left\{\int_0^1 \lambda^{p-\nu-1}(1-\lambda)^{\nu-1}s^{-(p-\nu)}W(\lambda/s)\,d\lambda\right\}_{s\to\xi}. \tag{2.14}$$

An application of Lemma 2.2 with $\alpha=1$ followed by the use of (2.5) with t replaced by $2\sqrt{\xi}$ and b replaced by $2p-1$ shows that the first member of (2.14) reduces to $\xi^{p-\nu-1}E^{2p-1}(2\xi^{1/2})/\Gamma(p)$. On the other hand, if we apply Lemma 2.4 to the second member of (2.14) and then successively apply Lemmas 2.2 (with $\alpha=\lambda$) and (2.5) with t replaced by $2\sqrt{\xi}$ and b replaced by $2p-2\nu-1$, the second member of (2.14) becomes

$$\frac{1}{\Gamma(\nu)\,\Gamma(p-\nu)}\xi^{p-\nu-1}$$
$$\times\int_0^1 \lambda^{p-\nu-1}(1-\lambda)^{\nu-1}E^{2p-2\nu-1}(2\sqrt{\lambda\xi})\,d\lambda.$$

The equality of these two terms follows by applying the parameter shifting formula

$$\mathscr{E}^{a_1+a_2}(t) = \frac{2t^{1-a_1-a_2}}{B\left(\frac{a_1+1}{2}, \frac{a_2}{2}\right)} \times \int_0^t (t^2-\eta^2)^{(a_2-2)/2} \eta^{a_1} \mathscr{E}^{a_1}(\eta)\, d\eta$$

[2, Theorem 2.1]. To see this, take $a_1 = 2p - 2\nu - 1$, $a_2 = 2\nu$, $t^2 = 4\xi$, and make the change of variables $\eta^2 = t^2\lambda$. From the fact that $\|T(t)\| \leqslant M_\omega e^{\omega t}$ for positive constants ω and M_ω [9, p. 306], the conditions of Lemma 2.4 (with appended note) are seen to be satisfied by the second member of (2.14). The choice of c depends upon ω.

Note 2. One can give a less complicated proof of this lemma by evaluating a complex contour integral. However, such a proof imposes unnecessarily strong restrictions on the semi-group and its generator.

Next, let $f(\xi)$ take on values in X, let $F(\mathfrak{s})$ denote its Laplace transform ($\mathfrak{s}$ positive), and let m be a positive integer. From the standard properties of the transform, it follows that

$$\int_0^\infty \xi^{-m} f(\xi)\, e^{-\mathfrak{s}\xi}\, d\xi = \int_{\mathfrak{s}}^\infty \frac{(y-\mathfrak{s})^{m-1}}{(m-1)!} F(y)\, dy$$

if either of the integrals in this exists. With the change of variables $y = \mathfrak{s}/\sigma$, we can write this property in the form

$$\int_0^\infty \xi^{-m} f(\xi)\, e^{-\mathfrak{s}\xi}\, d\xi = \frac{\mathfrak{s}^m}{(m-1)!} \int_0^1 \frac{(1-\sigma)^{m-1}}{\sigma^{m+1}} F(\mathfrak{s}/\sigma)\, d\sigma. \tag{2.15}$$

Finally, for the purpose of identifying certain inverse Laplace transforms in Sections 4 and 5 rewrite (2.5) in the form

$$\mathscr{L}_{\mathfrak{s}}^{-1}\{\mathfrak{s}^{-(c+1)/2} W(1/4\mathfrak{s})\}_{\mathfrak{s}\to R^2} = \frac{R^{c-1} E^c(R; \psi)}{\Gamma\left(\frac{c+1}{2}\right)}. \tag{2.16}$$

3. Solution Representations

With this background, we now give the solution representations for $U(t; \psi)$.

THEOREM 3.1. *Let $(a-b)/2 = m$, $m = 0, 1, 2, \dots$. Then*

$$U(t;\psi) = \Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{b+1}{2}+m\right)$$

$$\cdot \sum_{k=0}^{m} \binom{m}{k} \frac{(t/2)^{2m-2k+2}}{(m-k+1)\,\Gamma\left(\frac{b+1}{2}+m-k\right)\Gamma\left(\frac{a+3}{2}+m-k\right)}$$

$$\times A^{m-k} \cdot E^{a+2(m-k+1)}(t;\psi).$$

THEOREM 3.2. *Let $(a-b)/2 = m - \nu$ with $m = 0, 1, 2, \dots$ and $0 < \nu < 1$. Then*

$$U(t;\psi) = \frac{\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{b+1}{2}+m\right)}{\Gamma(\nu)}$$

$$\cdot \left[\sum_{k=0}^{m} \binom{m}{k} \frac{(t/2)^{2m-2k+2}}{\Gamma\left(\frac{b+1}{2}+m-k\right)\Gamma\left(\frac{a+3}{2}+m-k\right)} A^{m-k}\right.$$

$$\cdot \int_0^1 \int_0^1 (1-\lambda)^{\nu-1}\lambda^{(a-1)/2+m-k}$$

$$\left.\cdot\, \tau^{m-k}E^{a+2(m-k+1)}(t\sqrt{\lambda\tau+1-\tau};\psi)\,d\tau\,d\lambda\right].$$

THEOREM 3.3. *Let $(a-b)/2 = -m$, $m = 1, 2, \dots$. If $b > 2m-1$, then*

$$U(t;\psi) = \frac{\Gamma\left(\frac{b+1}{2}\right)t^2}{2(m-1)!\,\Gamma\left(\frac{a+3}{2}\right)}$$

$$\cdot \int_0^1 \int_0^1 \tau^{(b-1)/2-m}(1-\tau)^{m-1}E^{a+2}(t\sqrt{\lambda\tau+1-\lambda};\psi)\,d\lambda\,d\tau.$$

THEOREM 3.4. *Let $(a-b)/2 = -m-\nu$, $m = 1, 2, \dots$ and $0 < \nu < 1$. If $b > 2m+2\nu-1$, then*

$$U(t;\psi) = \frac{\Gamma\left(\frac{b+1}{2}\right)t^2}{4(m-1)!\,\Gamma(\nu)\,\Gamma\left(\frac{a+3}{2}\right)}$$

$$\cdot \int_0^1 \int_0^1 \int_0^1 \lambda^{(b-1)/2-m-v}(1-\lambda)^{m-1}(1-\tau)^{v-1}$$

$$\cdot \tau^{(b-1)/2-v} E^{a+2}(t\sqrt{\lambda\tau\sigma+1-\sigma};\psi)\, d\sigma\, d\lambda\, d\tau.$$

4. Proof of Theorem 3.1

Given that $(a-b)/2 = m$, the integral in the second member of (2.11) has the value $(-D_s)^m\{s^{-(b+1)/2}W(1/s)\}$. Then Lemma 2.1 with $\beta = (b+1)/2$ gives

$$f(2s^{-1/2}) = \frac{\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)} \sum_{k=0}^{m} \binom{m}{k}$$

$$\times \frac{\Gamma(\beta+m)}{\Gamma(\beta+m-k)}\, s^{-(m+1-k)} A^{m-k} W(1/s). \tag{4.1}$$

From this and strong Riemann integrability we see that it suffices to determine the contributions of the individual terms $f_k(2s^{-1/2}) = s^{-(m+1-k)}W(1/s)$ to the U function and then combine these with the other terms in the right member of (4.1) to obtain $U(t;\psi)$.

With this $f_k(2s^{-1/2})$ term, Eq. (2.8) for the corresponding function $Z_k(2\xi^{1/2})$ becomes

$$(D_\xi - A)\, Z_k(2\xi^{1/2}) = \xi^{m+1-k} W(\xi)$$

and, by Lemma 2.3, this has the solution $Z_k(2\xi^{1/2}) = (m-k+1)^{-1}\xi^{m-k+1}W(\xi)$. Using (2.9) followed by Lemma 2.2, the corresponding U_k is given by

$$U_k(2\xi^{1/2};\psi) = (m-k+1)^{-1}\xi^{(1-a)/2}\Gamma\left(\frac{a+1}{2}\right)$$

$$\times \mathscr{L}_s^{-1}\{s^{-(a+1+2m-2k+2)/2}W(1/s)\}_{s\to\xi}$$

$$= (m-k+1)^{-1}\xi^{(1-a)/2}\Gamma\left(\frac{a+1}{2}\right) 4^{-((a+1)/2+m-k)}$$

$$\times \mathscr{L}_s^{-1}\{s^{-(a+1+2m-2k+2)/2}W(1/4s)\}_{s\to 4\xi}. \tag{4.2}$$

We compare the inverse Laplace transform in the last member of (4.2) with formula (2.16). Making the choices $R = 2\xi^{1/2}$ and $c = a + 2m - 2k + 2$, (4.2) reduces to

$$U_k(2\xi^{1/2}; \psi) = \frac{(m-k+1)^{-1}\Gamma\left(\frac{a+1}{2}\right)\xi^{m-k+1}}{\Gamma\left(\frac{a+3}{2}+m-k\right)} \times E^{a+2(m-k+1)}(2\xi^{1/2}; \psi).$$

The representation stated in Theorem 3.1 follows by replacing ξ by $t^2/4$ in $U_k(2\xi^{1/2}; \psi)$ and then replacing the term $s^{-(m-k+1)}W(1/s)$ in the right member of (4.1) by $U_k(t; \psi)$ and β by $(b+1)/2$.

5. Proof of Theorem 3.2

For m a non-negative integer, we have

$$\begin{aligned} &\xi^m \mathscr{L}_s^{-1}\{s^{-\beta}W(1/s)\}_{s\to\xi} \\ &\quad = \mathscr{L}_s^{-1}\{(-D)^m[s^{-\beta}W(1/s)]\}_{\to\xi} \\ &\quad = \sum_{k=0}^{m}\binom{m}{k}\frac{\Gamma(\beta+m)}{\Gamma(\beta+m-k)}\mathscr{L}_s^{-1}\{s^{-(\beta+2m-k)}A^{m-k}W(1/s)\}_{s\to\xi}, \end{aligned} \tag{5.1}$$

$\beta = (b+1)/2$, in which the second member follows by a standard property of the Laplace tansform and the third member follows by an application of Lemma 2.1. The use of the condition on $(a-b)$ in (2.11) and strong Riemann integrability gives

$$\begin{aligned} f(2s^{-1/2}) = {} & \frac{\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)}\sum_{k=0}^{m}\binom{m}{k}\frac{\Gamma(\beta+m)}{\Gamma(\beta+m-k)}A^{m-k} \\ & \cdot\left\{s^{(a-1)/2}\int_0^\infty e^{-s\xi}\xi^{-\nu}\mathscr{L}_s^{-1}[s^{-(\beta+2m-k)}W(1/s)]_{s\to\xi}\,d\xi\right\}. \end{aligned} \tag{5.2}$$

It is evident from this that we need only determine the contribution to $U(t; \psi)$ of the bracketed term in (5.2). If we denote this by $f_k^*(2s^{-1/2})$, then Lemma 2.5 with $p = \beta + 2m - k$ yields

$$f_k^*(2s^{-1/2}) = \frac{1}{\Gamma(\nu)}\int_0^1 \lambda^{\beta+2m-k-\nu-1}(1-\lambda)^{\nu-1} \times s^{-(m-k+1)}W(\lambda/s)\,d\lambda.$$

The differential equation (2.8) for the corresponding function $Z_k^*(2\xi^{1/2})$ becomes

$$(D_\xi - A)Z_k^* = \frac{1}{\Gamma(\nu)} \int_0^1 \lambda^{\beta+2m-k-\nu-1}(1-\lambda)^{\nu-1}\xi^{m-k}W(\lambda\xi)\,d\xi$$

and, by Lemma 1.3, this has the solution

$$\begin{aligned} Z_k^*(2\xi^{1/2}) &= \frac{1}{\Gamma(\nu)} \int_0^1 \lambda^{\beta+2m-k-\nu-1}(1-\lambda)^{\nu-1} \\ &\quad \times \left\{ \int_0^\xi \eta^{m-k} W(\lambda\eta + \xi - \eta)\,d\eta \right\} d\lambda \\ &= \frac{1}{\Gamma(\nu)} \int_0^1 \int_0^1 \lambda^{\beta+2m-k-\nu-1} \\ &\quad \times (1-\lambda)^{\nu-1}\tau^{m-k}\xi^{m-k-1} W(\xi[\lambda\tau + 1 - \tau])\,d\tau\,d\lambda \qquad (5.3) \end{aligned}$$

in which the last member is obtained from the second by introducing the change of variables of integration $\eta = \xi\tau$.

To recover the corresponding $U_k^*(2\xi^{1/2}; \psi)$ function, we make use of (2.9) and Lemma 2.4. We get

$$\begin{aligned} U_k^*(2\xi^{1/2}; \psi) &= \frac{\xi^{(1-a)/2}\Gamma\left(\frac{a+1}{2}\right)}{\Gamma(\nu)} \\ &\quad \cdot \int_0^1 \int_0^1 \lambda^{\beta+2m-k-\nu-1}(1-\lambda)^{\nu-1}\tau^{m-k} \\ &\quad \cdot \mathscr{L}_\sigma^{-1} \left\{ \sigma^{-(a+1+2m-2k+2)/2} W\left(\frac{\lambda\tau + 1 - \tau}{\sigma}\right) \right\}_{\sigma\to\xi} d\lambda\,d\tau. \qquad (5.4) \end{aligned}$$

But by Lemma 2.2 with $\alpha = \lambda\tau + 1 - \tau$, the inverse Laplace transform in (5.4) can be rewritten as

$$\begin{aligned} &[4(\lambda\tau + 1 - \tau)]^{-(a+1+2m-2k)/2} \mathscr{L}_\sigma^{-1} \\ &\qquad \times \left\{ \sigma^{(a+1+2m-2k+2)/2} W\left(\frac{1}{\sigma}\right) \right\}_{\sigma\to 4(\lambda\tau+1-\tau)\xi}. \end{aligned}$$

Appealing to (2.16) with $c = a + 2m - 2k + 2$ and $R = 2(\lambda\tau + 1 - \tau)^{1/2}\xi^{1/2}$, this last term reduces to

$$\xi^{(a+1+2m-2k)/2} E^{a+2(m-k+1)}(2(\lambda\tau + 1 - \tau)^{1/2}\xi^{1/2}).$$

Replacing the inverse transform in (5.4) by this last expression, we obtain

$$U_k^*(2\xi^{1/2};\psi)=\frac{\Gamma\left(\dfrac{a+1}{2}\right)}{\Gamma(\nu)}\xi^{m-k}$$

$$\cdot\int_0^1\int_0^1 \lambda^{\beta+2m-k-\nu-1}(1-\lambda)^{\nu-1}\tau^{m-k}E^{a+2(m-k+1)}$$

$$\cdot\,(2\sqrt{(\lambda\tau+1-\tau)\xi})\,d\lambda\,d\tau.$$

The validity of the representation formula in Theorem 3.2 follows by (i) replacing ξ by $t^2/4$ in $U^*(2\xi^{1/2};\psi)$; (ii) replacing β by $(b+1)/2$ and by $(a+1)/2-m+\nu$ in the exponent of λ, and (iii) replacing the bracketed term in (5.2) by this $U_k^*(t;\psi)$ term.

6. Remarks on the Other Theorems.

We conclude this paper by determining the functions $f(2\sigma^{-1/2})$ that occur in the derivations of the formulas in Theorems 3.3 and 3.4. The techniques that were applied to obtain the $U_k^*(t;\psi)$ terms in Section 5 can then be used with little modification to recover the corresponding solution functions $U(t;\psi)$.

(i) $(a-b)/2=-m$, $m=1,2,\ldots$. In this case, we obtain $f(2\sigma^{-1/2})$ in (2.11) by using (2.15) with $F(\sigma)=\sigma^{-(b+1)/2}W(1/\sigma)$. Then

$$f(2\sigma^{-1/2})=\frac{\Gamma\left(\dfrac{b+1}{2}\right)}{(m-1)!\,\Gamma\left(\dfrac{a+1}{2}\right)}$$

$$\times\int_0^1 \tau^{(b-1)/2-m}(1-\tau)^{m-1}\sigma^{-1}W(\tau/\sigma)\,d\tau.$$

The inequality on b stated in the theorem is needed to ensure the existence of this integral.

(ii) $(a-b)/2=-m-\nu$. With the choice $f(\xi)=\xi^{-\nu}\mathscr{L}_\sigma^{-1}\times\{\sigma^{-(b+1)/2}W(1/\sigma)\}_{\sigma\to\xi}$ in (2.15), we get

$$f(2s^{-1/2}) = \frac{\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)(m-1)!}\, s^{(a-1)/2}$$

$$\times \left\{ s^m \int_0^1 \frac{(1-\lambda)^{m-1}}{\lambda^{m+1}} F(s/\lambda)\, d\lambda \right\}.$$

Using Lemmas 2.5 to rewrite $F(s/\lambda)$ as an integral involving the function W, this becomes

$$f(2s^{-1/2}) = \frac{\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma(\nu)(m-1)!}$$

$$\cdot \int_0^1 \int_0^1 (1-\lambda)^{m-1} \lambda^{(b-1)/2-m-\nu} \tau^{(b-1)/2-\nu}$$

$$\cdot (1-\tau)^{\nu-1} W\left(\frac{\tau\lambda}{s}\right) d\tau\, d\lambda.$$

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